Congruence arithmetic.

Lemma: If x and y are multiples of n, so is $x + y$.

Proposition: Given integers a, b, c, and a natural number n, if$a \equiv b \pmod{n}$ then

- If $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$
- $ac \equiv bc \pmod{n}$

Definition: x, y, n are integers, $n \neq$

then $\boxed{x \equiv y \pmod{n}}$

if and only if $n \mid (x-y)$

- $x \equiv 0 \bmod 2$ means $2 \mid x \iff x$ is even
- $x \equiv 1 \bmod 2$ means $2 \mid (x-1)$

means $x-1 = 2k$ for some $k \in \mathbb{Z}$

$x = 2k+1$ for some $k \in \mathbb{Z}$

- $x \equiv 5 \bmod 2$ means $2 \mid (x-5)$

$x = 2t+5$

$= 2t + 2\cdot 2 + 1$

$= 2(t+2)+1$

$k = t+2$

$x - 5 = 2t \quad t \in \mathbb{Z}$

$x = 2t+5 \quad t \in \mathbb{Z}$

$x = 2(k-2)+5$

$= 2k-4+5 = 2k+1$

$\{x : x \equiv a \bmod n\}$

$x = a + kn$

$\{\ldots, a-2n, a-n, a, a+n, a+2n, \ldots\}$.

arithmetic progression

congruence class

$a \equiv 2 \ (5)$

$\ldots, -8, -3, 2, 7, 12, 17, 22$

Division algorithm

Given a, b, there are q, r so that

$$\boxed{b = qa + r}$$

$$0 \leq r < a$$

a divides $b - r$

$$b \equiv r \bmod a \qquad 0 \leq r < a$$

$$a = 2$$

$$a \equiv \begin{cases} 0 \bmod 2 \\ 1 \bmod 2 \end{cases} \quad r = 0, 1$$

$$a \equiv \begin{cases} 0 \\ 1 \\ 2 \end{cases} \bmod 3$$

$$5 \equiv 2 \ (3)$$

$$-2 \equiv 1 \ (3)$$

$$6 \equiv 0 \ (3)$$

x even

$x \equiv 0 \pmod 2$

y odd

$y \equiv 1 \pmod 2$

$x + y \equiv 0 + 1 \mod 2$

$\equiv 1 \mod 2$

$x \equiv 1 \mod 2$

$y \equiv 1 \mod 2$

odd + odd

$\Downarrow$

even

$x + y \equiv 2 \mod 2$

$2 \equiv 0 \mod 2$

$x + y \equiv 0 \mod 2$

Prop: If $x \equiv a \mod n$, $y \equiv b \mod n$, then $x + y \equiv a + b \mod n$

Given $x \equiv a \mod n$, then $n \mid (x - a)$

$y \equiv b \mod n$ then $n \mid (y - b)$

Since $n \mid (x - a)$, there is a $k_1 \in \mathbb{Z}$ so that

$x - a = k_1 n$ or $x = k_1 n + a$

$n \mid (y - b)$, there is $k_2 \in \mathbb{Z}$ so that

$y - b = k_2 n$

Goal: Show $n \mid (x+y-a-b)$

have $\underbrace{(x-a)+(y-b)}_{x+y-a-b} = k_1 n + k_2 n$

$= (k_1+k_2)n$

so $n \mid (x+y-a-b)$

Suppose $x \equiv a \bmod n$

$y \equiv b \bmod n$

Show $xy \equiv ab \bmod n$

Given $x-a = k_1 n$

$y-b = k_2 n$

$x = k_1 n + a \qquad y = k_2 n + b$

$xy = (k_1 n + a)(k_2 n + b)$

$= k_1 k_2 n^2 + k_1 b n + k_2 a n + ab$

$xy = n(k_1 k_2 n + k_1 b + k_2 a) + ab$

Congruence arithmetic continued

Proposition: If $a \equiv b \bmod n$, then

$a^r \equiv b^r \pmod{n}$ for any natural number r.

$$\begin{cases} a \equiv b \bmod n \\ a^2 \equiv b^2 \bmod n \\ a^3 \equiv b^3 \bmod n \\ \vdots \end{cases}$$

Casting out nines

111

54333

Sum of digits div by 3 → # div by 3

9 → # div by 9

54333

$3 + 3 \cdot 10 + 3 \cdot 10^2 + 4 \cdot 10^3 + 5 \cdot 10^4$

$10 \equiv 1 \mod 9$

$10^2 \equiv 1 \mod 9$

$10^3 \equiv 1 \mod 9$

$10^4 \equiv 1 \mod 9$

$3 + 3 \cdot 10 + 3 \cdot 10^2 + 4 \cdot 10^3 + 5 \cdot 10^4 \equiv$

$3 + 3\cdot 1 + 3\cdot 1 + 4\cdot 1 + 5\cdot 1 = 18 \mod 9$

$\equiv 0 \mod 9$

~~[illegible]~~

$10 \equiv -1 \mod 11$

even powers of $10 \equiv 1 \mod 11$ odd powers $\equiv -1 \mod 11$

$3 + 3\cdot 10 + 3\cdot 10^2 + 4\cdot 10^3 + 5\cdot 10^4$

$3 - 3 + 3 - 4 + 5$

$54333 = 4 \mod 11$

$10^k \equiv 0 \mod 8$ once $k \geq 3$

54333

$10 \equiv 3 \quad (7)$

$100 \equiv 9 \equiv 2 \quad (7)$

$1000 = 6 \quad (7)$

$\equiv -1$

-3

-2